

GROUP AVERAGING AND REFINED ALGEBRAIC QUANTIZATION: WHERE ARE WE NOW?

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Refined Algebraic Quantization and Group Averaging are powerful methods for quantizing constrained systems. They give constructive algorithms for generating observables and the physical inner product. This work outlines the current status of these ideas with an eye toward quantum gravity. The main goal is provide a description of outstanding problems and possible research topics in the field.

1 Introduction

The well-known connection between gauge symmetries and first class constraints means that constrained systems are ubiquitous in modern physics. Studying quantization of such systems dates back to Dirac¹, who described a general procedure. However, this ‘procedure’ is far from algorithmic and leaves open many questions. While in practice there is little difficulty for simple systems, issues such as the construction of both observables and the physical inner product take on a qualitatively different character for more complicated systems, such as gravity². In the gravitating case, these issues are associated with the famous ‘problem of time.’ Thus, an interest in canonical quantum gravity^{2,3} forces one to reexamine various subtleties. We remind the reader that Dirac style methods are common both in the proposed loop representation^{3,4} for full quantum gravity and in quantum cosmology, whether inspired by Einstein-Hilbert gravity^{5,6} or string/M-theory^{8,9}.

Refined algebraic quantization^{10,11} and related work^{12,13,14} form an on-going program to resolve the subtleties of Dirac quantization. It is closely associated with a technique known as ‘group averaging.’ In particular it was recently shown¹⁵ that, when it is well-defined, group averaging gives the unique implementation of refined algebraic quantization. A strength of this approach is that it gives a constructive method for obtaining both the physical inner product and interesting observables. Other programs to refine Dirac’s scheme include geometric quantization^{16,17,18}, BRST^{19,20} methods, Klein-Gordon methods⁶, coherent state quantization²¹, C*-algebra methods²², and algebraic quantization³.

The goal of this work is to outline the current state of refined algebraic quantization with emphasis on open questions and possible research projects. First, however, we give a brief review of the method. More complete introductions and reference lists are contained in other works^{10,11,7}. While we do point out important technical caveats, we will avoid going into them in detail.

2 Review of group averaging and refined algebraic quantization

As our procedures are based on that of Dirac, we discuss two spaces of states. The first is an auxiliary or kinematical space, on which the constraints are to be formulated as operators. The goal of the Dirac scheme is to construct a second

space of ‘physical’ states $|\psi\rangle_{\text{phys}}$ which in some sense solve the constraints.

The physical states must then be made into a Hilbert space. To do so requires an inner product, which can be particularly obscure. Indeed, when the constraints cannot be solved exactly, one might wonder how such an inner product could be constructed. A natural idea³ is to relate it to properties of the observable algebra. However, this algebra also typically cannot be found in closed form.

The idea of refined algebraic quantization is to relate construction of the physical inner product to a much easier problem. Dirac’s space of auxiliary (or kinematical) states is in some sense a quantization of a corresponding unconstrained system, as one simply ignores the constraints at this stage. This system can be quantized as usual, without subtleties stemming from constraints. Thus, if one wished, one make the kinematical states into a Hilbert space \mathcal{H}_{kin} by supplying an inner product. In particular, one may choose the inner product on this space to map certain real functions on the unconstrained phase space to Hermitian operators on \mathcal{H}_{kin} .

Now, given the Hilbert space \mathcal{H}_{kin} , what group averaging supplies is a constructive method for solving the constraints and defining a ‘physical’ inner product on the solutions. This inner product captures the reality conditions of observables in the following sense: Given a Hermitian operator \mathcal{O} on \mathcal{H}_{kin} that commutes with the constraints, it defines a Hermitian operator on the physical Hilbert space. Since the classical reality conditions are already encoded in the inner product on \mathcal{H}_{kin} , this process also implements them on the physical Hilbert space. In addition, group averaging gives a method of first constructing these observables on \mathcal{H}_{kin} .

Let us turn to the details of these constructions. It is sufficient to consider the case where the classical constraints are real. Furthermore, for the moment we take the constraints to form the Lie algebra of a unimodular Lie group G . More general Lie groups will be discussed shortly. We note however that the hypersurface deformation algebra of canonical gravity does not form a Lie algebra at all. Instead of structure constants, this algebra contains ‘structure functions.’ Whether group averaging can be generalized to accommodate structure functions is an interesting research question to which considerable discussion will be devoted below.

Since the classical constraints are real, it is natural to implement them on \mathcal{H}_{kin} as Hermitian operators^a which then generate a unitary representation $U(g \mapsto U(g))$ of the gauge group G . Dirac would ask that the physical states solve the constraint in the sense that they be annihilated by the constraints. The same requirement is given by the statement that the unitary operators $U(g)$ should act trivially on the physical states for any g in the gauge group:

$$U(g)|\psi\rangle_{\text{phys}} = |\psi\rangle_{\text{phys}}. \quad (1)$$

Now, as 1 need not lie in the discrete spectrum of $U(g)$, the Hilbert space \mathcal{H}_{kin} need not contain any such solutions. This is the typical behavior for non-compact groups. However, we may seek solutions in a suitable space $\Phi^* \supset \mathcal{H}_{\text{kin}}$ of ‘generalized states.’ One chooses a subspace $\Phi \subset \mathcal{H}_{\text{kin}}$ of ‘test states’ and then takes $\Phi^* \supset \mathcal{H}_{\text{kin}}$ to be the space of all linear functionals on Φ that are continuous with respect to the topology induced from \mathcal{H}_{kin} . The space Φ should be chosen so that the operators $U(g)$ map

^aThis is in general not possible when the constraint algebra contains structure functions²³, thus the obstacle to treating structure functions with group averaging.

Φ into itself. In this case, there is a well-defined dual action of $U(g)$ on $f \in \Phi^*$ given by $[U(g)f](\phi) = f(U(g^{-1})\phi)$ for all $\phi \in \Phi$. Solutions of the constraints are then elements $f \in \Phi^*$ for which $U(g)f = f$ for all g .

Consider for the moment the case of compact groups. Then \mathcal{H}_{kin} can be decomposed as a direct sum of irreducible representations of G . In this case, the integral $\int_G dg U(g)$ gives the operator P_0 that projects onto states in the trivial representation of G . In other words, this operator projects onto just the set of states that solve the constraint. Here, dg is the (unique) Haar measure on the group.

Let us now consider the integral $\int_G dg U(g)$ for arbitrary unimodular Lie groups. In general, Lie groups have two Haar measures: one invariant under left translations ($g \rightarrow g_0 g$) and one invariant under right translations ($g \rightarrow g g_0$). Unimodular groups are those groups for which these Haar measures coincide. In this case, the (unique) Haar measure is also invariant under the map $g \rightarrow g^{-1}$. All compact groups are unimodular, as are many non-compact groups.

When the group is not compact, $\int_G dg U(g)$ will not converge as an operator on \mathcal{H}_{kin} and so will not define a projector. However, physical states need not actually lie in \mathcal{H}_{kin} . Thus, group averaging might still allow us to ‘project’ states (at least those lying in Φ) onto solutions lying in Φ^* . The map $\eta : \Phi \rightarrow \Phi^*$, $\phi \mapsto \langle \phi | \int_G dg U(g)$ is well-defined when the integral $\int_G dg \langle \alpha | U(g) | \beta \rangle$ converges for all α, β in Φ . Let us assume that this is so. Note that η is anti-linear as is natural for a map from a space (Φ) to its dual (Φ^*). Translation invariance of the Haar measure dg guarantees that $\eta(\phi)$ solves the constraints in the sense described above.

Group averaging uses the image of η and the inner product $(\eta(\alpha), \eta(\beta))_{phys} = \eta(\beta)[\alpha]$ on this space to define the physical Hilbert. The reversal of positions of α, β on the left and right is due to the anti-linearity of η . That this inner product is Hermitian follows from the invariance of dg under $g \mapsto g^{-1}$. If this is to be a valid physical inner product, it must also be positive definite. This is true in every case known to the author, but has not been established in complete generality. We will return to the positivity issue below in our discussion of open questions.

The physical inner product has an important property advertised above: Hermitian gauge invariant observables on \mathcal{H}_{kin} will become Hermitian operators on the physical Hilbert space. In refined algebraic quantization, observables are required to include Φ in their domain and to map Φ to itself. Such operators act on states \tilde{phi} in the dual space Φ^* through $(\mathcal{O}\tilde{\phi})[\alpha] = \tilde{\phi}(\mathcal{O}^\dagger \alpha)$. ‘Gauge invariance’ of such an operator \mathcal{O} then means that \mathcal{O} commutes with the G -action on the domain Φ : $\mathcal{O}U(g)|\phi\rangle = U(g)\mathcal{O}|\phi\rangle$ for all $g \in G$, $\phi \in \Phi$. Any such observable will commute with the group averaging map η , in the sense that $\mathcal{O}\eta(\phi) = \eta(\mathcal{O}\phi)$. This in turn means that any relation of the form $A = B^\dagger$ between two observables A and B on \mathcal{H}_{kin} also holds between the corresponding observables on \mathcal{H}_{phys} . This property is what makes the group averaging inner product physically interesting.

Refined algebraic quantization generalizes this idea to allow any map η (called the ‘rigging map’) having the properties deduced for group averaging above. The choice of this map appears to be an extra degree of freedom. Φ has been chosen. One can then show If the group averaging map converges on Φ and yields a nontrivial result, then¹⁵ (up to an overall scale) it is in fact the unique rigging map.

This completes the construction for the unimodular case, but what of the non-

unimodular case? Now there are two Haar measures: a left measure $d_L g$ invariant under $g \mapsto g_0 g$ and a right measure $d_R g$ invariant under $g \mapsto g g_0$. Neither is invariant under $g \mapsto g^{-1}$. Convergence of group averaging (in both measures) now implies¹⁵ that *no* rigging map exists which i) is Hermitian, ii) commutes with the observables, and iii) solves the constraints in the sense that $\eta(U(g)\phi) = \eta(\phi)$.

However, this is not the end of the story. One can use consistency of group averaging¹⁵ or geometric quantization¹⁸ to argue that, for non-unimodular groups, the physical states should not in fact be annihilated by the constraints. Instead, one finds that physical states ψ_{phys} should satisfy $U(g)\psi_{phys} = \Delta(g)\tilde{\psi}_{phys}$, where $\Delta(g)$ is the so-called ‘modular function’ on the group, a one-dimensional non-unitary representation of the group, $\Delta : G \rightarrow \mathbf{R}^+$ satisfying $d_R g = \Delta(g)d_L g$. In terms of the Hermitian generators C_i , this means that physical states satisfy

$$C_i|\psi\rangle_{phys} = \frac{i}{2}tr_{ad}(C_i)|\psi\rangle_{phys}, \quad (2)$$

instead of being annihilated by the constraint. Here tr_{ad} denotes the trace in the adjoint representation. Since C_i is Hermitian, $itr_{ad}(C_i)$ cannot lie in its spectrum unless $tr_{ad}(C_i)$ vanishes. Nonetheless, generalized states $\psi_{phys} \in \Phi^*$ satisfying (2) may exist. With the understanding (2), convergence of group averaging guarantees¹⁵ that $\phi \mapsto \langle \phi | \int_G dg_0 U(g)$ is the unique rigging map. Here, $d_0 g = \Delta^{1/2} d_L g = \Delta^{-1/2} d_R g$ is in fact invariant under $g \mapsto g^{-1}$.

Before moving on to open questions, we comment briefly on constructing observables. Just as group averaging constructs a physical state $\eta(\phi)$ from a kinematical state ϕ , so it also constructs observables (acting on \mathcal{H}_{phys}) from operators on \mathcal{H}_{kin} (‘kinematical observables’). The idea is simple. Given an operator O on \mathcal{H}_{kin} and two states $\alpha, \beta \in \mathcal{H}_{kin}$ we may compute: $\int d_L g \langle \alpha | U(g) O U(g^{-1}) | \beta \rangle$. If this expression converges sufficiently rapidly, it defines an observable. This observation was used^{14,24} in a minisuperspace context to construct observables of the ‘evolving constants of motion’ type^{25,26}. There, it was possible to work with such quantum observables (and even to compute their matrix elements), without being able to write the corresponding classical observables as explicit phase space functions.

3 Where are we now?

In specialized fields, it can be difficult for non-experts to locate the frontier of research. Newcomers and beginning graduate students are often unsure just which questions have been answered and which remain open. In order to assist such researchers, the rest of this work outlines open questions and possible directions for investigation in group averaging and refined algebraic quantization. Several of associated research projects are straightforward and some of them are at a level accessible to beginning graduate students. It is hoped that this list will encourage a broader range of physicists and mathematicians to contribute to the field.

3.1 Comparison with other methods

Direct application of the Dirac quantization scheme has been a favorite approach to constrained system quantization within much of the general relativity commu-

nity. However, physicists in other fields are more familiar with other techniques for quantizing gauge systems. Primary among these are the Fadeev-Popov method and its extension through BRST¹⁹ quantization. A careful comparison of group averaging techniques with these methods is long overdue. While at some rough level it is known that all of these techniques are equivalent to Dirac quantization, the application of group averaging represents a refinement of the Dirac idea and one would like to compare the approaches at this finer level of detail. For example, recalling that these other schemes are often discussed within a path integral framework and assuming that there is an equivalence, one would like to know just which of choice of measure makes the procedures agree.

A few results relating group averaging to other approaches are known. First consider the usual minisuperspace setting, with a single constraint and a Hamiltonian that vanishes on the constraint surface. Such systems may be explored²⁸ by writing the $U(g)$ that appears in the group averaging expression as a path integral. In that context, one can show that a path integral of the usual Fadeev-Popov form computes exactly the matrix elements of the group averaging map η . That is to say that, given states α and β as the boundary conditions of the path integral, the path integral computes $\eta(\beta)[\alpha]$. By the usual reasoning, this path integral can be written in the BRST form as well. One obtains a specific measure and range of integration over the Lagrange multipliers. Some affects of these details may be seen through semiclassical approximation. In particular²⁸, they affect how instantons with negative Euclidean action should contribute. The conclusion is that Euclidean instantons contribute to matrix elements of the rigging map as $\exp(-|S_E|)$, with a corresponding result for complex instantons.

Another straightforward case occurs when the gauge group is unimodular and the constraints are linear in momenta. Note that the term unimodular necessarily implies that we consider a finite dimensional Lie group, and in particular that we are in a quantum mechanical setting as opposed to that of a local field theory. In this case one can again introduce a path integral expression for the rigging map and show that matrix elements of the form $(e^{iHt}\eta(\beta))[\alpha]$ take the form of a standard Fadeev-Popov path integral. One can also show directly using operator methods that group averaging, meaning a scheme using both the group averaging inner product and a construction of observables through group averaging, is equivalent to a scheme in which one first fixes a (coordinate) gauge and then quantizes the system. This is closely related to the observations of Woodard²⁷. The particularly simple case of the parametrized particle is treated in the literature,¹⁴ but the general case of this sort is quite similar.

We see, however, that there are ways in which additional subtleties can enter the story. The first is the case of non-unimodular groups. Due to the usual emphasis on compact (local) gauge groups, these are not typically considered in the framework of Fadeev-Popov and BRST. Based on the group averaging¹⁵ and geometric quantization results¹⁸ described above, one suspects that Fadeev-Popov and BRST techniques require a small adjustment in non-unimodular case. After this is done, one expects to again find agreement with group averaging for constraints linear in momenta though this has not yet been shown.

The second place where subtleties may arise is for constraints not linear in mo-

menta. Let us recall first how this complicates the Fadeev-Popov framework itself. Suppose that one works in the canonical setting. Because the constraints are non-linear in momenta, the Fadeev-Popov determinant will typically involve momenta as well. Thus the fully gauge-fixed action is no longer precisely quadratic in momenta and the translation back and forth between (gauge-fixed) Lagrangian and Hamiltonian settings is complicated. Naive application of Fadeev-Popov methods in the Lagrangian and Hamiltonian frameworks then leads to inequivalent results. More precisely, choosing a simple measure for the path integral on one side may correspond to choosing a complicated measure on the other side. It is thus of interest to find out what prescription if any corresponds to group averaging. Because group averaging is essentially a canonical scheme, one would expect the measure to appear simple in terms of a canonical path integral. This is just what was found in²⁸ for the case of a single constraint, but the general case remains to be done. It would also be interesting to revisit Woodard's arguments²⁷ with this in mind.

A field theory setting might provide a third source of subtleties. Rigorous results for this case may be difficult, but even a heuristic treatment may prove useful. Finally, one would like to investigate further the relation between group averaging and geometric quantization^{16,17,18}, coherent state quantization²¹, C*-algebra methods²², and Klein-Gordon methods⁶ (in the context of constraints quadratic in momenta). Some remarks on the relation to Klein-Gordon methods have appeared^{29,30,14}. These point out that one may use the relation between group averaging and Klein-Gordon methods to define a vacuum state on curved spacetimes, though it is not clear what physical status this vacuum might have.

3.2 Structure Functions

Perhaps the most interesting open question is whether group averaging can be generalized to systems with structure functions. Since structure functions appear in the hypersurface deformation algebra of gravitating systems^b, this issue must be resolved before one may use group averaging for full quantum gravity.

While the leap to structure functions may seem like a large one, reasons for optimism exist. For example, recall²³ that the difficulty is tied to the current insistence on using Hermitian operators for the constraints. But perhaps the constraints could have an anti-Hermitian part? Returning to the case of non-unimodular Lie groups, we see that the answer is affirmative. Recall^{18,15} that quantization of non-unimodular Lie groups can proceed with Hermitian constraints C_i , but that physical states satisfy (2) instead of the more familiar condition that they are annihilated by the constraints. However, (2) is equivalent to considering constraints $\tilde{C}_i = C_i - itr_{ad}(C_i)$ which annihilate the constraints. One may check that the \tilde{C}_i generate a non-unitary representation of the same group as the constraints C_i . In fact, if C_i generate the representation $U(g)$, then \tilde{C}_i generate $\Delta^{-1}(g)U(g)$. Something similar should arise for structure functions.

Another useful observation is that one may convert an algebra with structure constants into an algebra with structure functions by merely multiplying each classical constraint by a function on phase space. A similar statement is true in the

^bThough they are absent in minisuperspace models and can be removed in the 1+1 case.

quantum setting if one multiplies the constraints on the left by operators. If the original constraints of the Lie algebra were Hermitian, the new constraints will in general not be so. One may hope to gain control over the structure function case by examining in detail such cases of ‘artificial’ structure functions where an honest Lie group lurks in the background. One would hope in this way to build on lessons from the non-unimodular case and come to terms with structure functions.

3.3 *When Group Averaging Fails*

We have said that group averaging and refined algebraic quantization can to a large extent be identified. This follows from a theorem¹⁵ which states that, when group averaging converges, it gives the unique implementation of refined algebraic quantization. However, what happens when group averaging fails to converge is an interesting and very open area of investigation.

Only a few examples have been studied: the diffeomorphism constraints of gravity in the loop representation¹⁰, cases involving a single constraint³¹, and the action of $SO(n,1)$ on functions on Minkowski space³². In all cases some ‘renormalization’ of the group averaging map was used. In particular, a subspace Φ of the kinematical Hilbert space split into different ‘sectors’ in which group averaging required different amounts of renormalization. These sectors were in fact superselected, in the sense that any operator which commutes with the constraints and preserves Φ has vanishing matrix elements between states in different sectors. This allows a freedom to rescale the physical inner product separately on each superselected sector. Refined algebraic quantization is no longer unique, though this non-uniqueness does not affect any physics. It is appealing to think that this is the general picture, but the evidence is too preliminary to draw firm conclusions. Further investigation of examples is needed, which one hopes will lead to general theorems.

3.4 *Study of Examples*

Much of the work to date in group averaging and refined algebraic quantization has centered on deriving general results and developing the overall structure of the approach. While this has resulted in powerful theorems, the methods are sufficiently abstract that it would be useful to have more concrete examples worked out in detail. A few examples were studied in early works^{14,24}, but even there the detailed answers to physical questions were not computed. The focus was on showing that both the physical inner product and operators of the evolving constant of motion type were well defined. Missing were computations of expectation values or matrix elements of such operators in particular physical states which might shed light on the physical implications of quantum gravity, and more detailed studies of the time evolution inherent in quantum mechanical evolving constants of motion. A semi-classical treatment³⁴ is useful to illustrate the sort of effects that may arise.

A particularly interesting case is the Bianchi IX model. This model has been a favorite of minisuperspace workers and is now thought³³ to be connected to the generic behavior of general relativity near a singularity. It has been argued using a WKB-like self-consistent approximation scheme that refined algebraic quantization can be applied to this model²⁴ though again without calculations of detailed

properties of observables. This complicated model may be an appropriate place to introduce numerical techniques, and the development of numerical techniques for group averaging is an interesting direction of research in itself.

In addition, it is by no means clear that the constructions of evolving constants of motion via group averaging used in the early works^{14,24} are the best. The goal there was to prove rigorously that a complete set of such observables exists. To do so, certain “regulators” were introduced. While these regulators have no effect in the classical limit, they look rather ad hoc from the quantum mechanical perspective. One may expect that such regulators are in fact not necessary, though it will take more detailed calculations to show this. It is exactly this sort of issue that might be best probed by starting with particular models and working out the details.

Another feature deserving of more insight is the $itr_{ad}(C_i)$ term required for non-unimodular groups. Equation (2) implies that the Hermitian constraints C_i do not annihilate physical states. As a result, one might think of physical states as having a certain finite ‘width’ around the constraint surface. Since, however, one may also use the non-Hermitian constraints \tilde{C}_i that do annihilate physical states, it is not clear whether this is a useful way in which to think about the effect. Perhaps one can gain more insight into by studying particular systems in detail.

3.5 Semiclassical Techniques

Another direction to explore is the use of semiclassical techniques. Some results about the group averaging map are known²⁸ to leading semi-classical order, $e^{iS/\hbar}$. However, an extension of this study to higher orders would be worthwhile. In particular, it is at the next order where one would expect to see contributions from the particular measure in a path integral associated with group averaging.

A traditional application of semiclassical techniques in quantum gravity has been to ‘the problem of time’ and further studies are warranted in the group averaging context. While much of the general theory for the minisuperspace case has been developed³⁴, one would like to see detailed investigations in particular models as well as a generalization to cases with multiple constraints.

3.6 Remaining Issues

A few issues remain that, while they have a rather technical flavor, may be of great importance. The first has to do with an ambiguity in the approach: When group averaging converges on a given space Φ , one finds¹⁵ that this process gives the unique rigging map η . However, this says nothing about how Φ is to be chosen in the first place and to what extent it is unique. An initial exploration³¹ indicated that Φ will not be determined by mathematical consistency alone. Instead, it was conjectured that the choice of Φ contains physical input related to the classical choice of differential structure on the unconstrained phase space. In particular, it would therefore encode the differential structure of the constraint surface. However, the details of this idea remain to be fleshed out in full.

Another important issue concerns the positivity of the group averaging inner product. Suppose that group averaging converges nontrivially. Then what is known is that any implementation of refined algebraic quantization is proportional to the

group averaging result. Thus, positivity of group averaging is closely linked to the success of refined algebraic quantization in constructing a positive definite physical Hilbert space. No cases are presently known in which group averaging fails to be positive definite, but this issue clearly deserves further study.

A final issue involves a connection with representation theory. Recall our discussion of group averaging for compact groups. There, it yielded a projector onto a subspace associated with the trivial representation in a decomposition of \mathcal{H}_{kin} into irreducibles. However, this is not the general case¹⁵ for group averaging. Indeed, for type II and type III groups³⁵ (also known as ‘wild’ groups³⁶), a unique decomposition into irreducibles may not exist. Now, the usual decomposition into irreducibles is associated with the mathematical notion of ‘weak containment’ of representations^{35,36}. One suggestion¹⁵ is that refined algebraic quantization may instead relate to a new ‘ultra-weak’ notion of containment. This should be explored both for the sake of mathematics and with an eye toward understanding cases where group averaging fails. As in other settings, a general understanding of the abstract structure of a procedure may allow one to find and control features which are difficult to grasp through detailed calculations.

The above ‘laundry list’ of issues shows both the variety of results obtained to date and the amenability of the field to further study. As in many cases, there is a need for both detailed investigation of particular models and abstract work on general principles. Such a field is open to a variety of researchers and it is difficult to predict from which corner the next important insights will emerge.

Acknowledgements

This work was supported in part by NSF grant PHY97-22362, the Alfred P. Sloan foundation, and funds from Syracuse University. The author wishes to thank A. Ashtekar, C. Fewster, A. Gomberoff, A. Higuchi, K. Kuchar, J. Lewandowski, J. Mourão, C. Rovelli, T. Thiemann, and of course D. Giulini for many useful conversations on refined algebraic quantization and group averaging.

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